

## ON A THEOREM OF CASTELNUOVO AND APPLICATIONS TO MODULI

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**ABSTRACT.** In this paper we prove a theorem stated by Castelnuovo in [7] which bounds the dimension of linear systems of plane curves in terms of two invariants, one of which is the genus of the curves in the system. This extends a previous result of Castelnuovo–Enriques (see [12]). We classify linear systems whose dimension belongs to certain intervals which naturally arise from Castelnuovo’s theorem. Then we make an application to the following moduli problem: what is the maximum number of moduli of curves of geometric genus  $g$  varying in a linear system on a surface? It turns out that, for  $g \geq 22$ , the answer is  $2g + 1$ , and it is attained by trigonal canonical curves varying on a balanced rational normal scroll.

## INTRODUCTION

This paper has been originated by the following problem (see Problem 2.1). Consider the set  $\mathcal{X}_g$ , with  $g \geq 2$ , of all linear systems  $\mathcal{L}$  of curves on a surface  $X$  such that the general curve of  $\mathcal{L}$  is irreducible, with geometric genus  $g$ . For such an  $\mathcal{L}$ , consider its image in  $\mathcal{M}_g$  via the obvious (rational) moduli map. What is the maximum dimension of this image (called the *number of moduli* of  $\mathcal{L}$ ) when  $\mathcal{L}$  varies in  $\mathcal{X}_g$ ?

A naive expectation is that, the larger the dimension of  $\mathcal{L}$ , the larger its number of moduli. So a related question is: what is the maximum of  $r = \dim(\mathcal{L})$  as  $\mathcal{L}$  varies in  $\mathcal{X}_g$ ? This has a classical answer which goes back to Castelnuovo [6] and Enriques [13]. They proved an important result (see Theorem 1.1) to the effect that  $r \leq 3g + 5$ , with three exceptions which, up to birational equivalence, are the following: either  $\mathcal{L}$  is the linear system of plane cubics or the rational map determined by  $\mathcal{L}$  realizes  $X$  as a scroll. In the former case the number of moduli is 0, in the latter it is 1. Castelnuovo and Enriques also classified the cases in which the bound  $r = 3g + 5$  is attained:  $X$  is then rational and  $\mathcal{L}$  is either (up to birational equivalence) the linear system of plane curves of degrees 2 or 4 or a suitable system of hyperelliptic curves. Castelnuovo–Enriques’ theorem has been rediscovered and/or reconsidered a few times in the course of the years: see [12] for classical and more recent references.

At about the same time, Castelnuovo stated in [7], with a rather sketchy proof, a more general and interesting theorem which classifies linear systems on rational surfaces with  $r > g$  (see Theorem 1.3). Castelnuovo’s argument is based on an ingenious application of adjunction and on a basic inequality (see Theorem 1.2) which improves the original Castelnuovo–Enriques theorem. Castelnuovo’s Theorem 1.3 is a very interesting result in birational geometry of surfaces, and more recent developments, e.g. [16, Corollary (1.1)], are reminiscent of it. Section 1 is devoted to prove, following and clarifying Castelnuovo’s original idea, Castelnuovo’s inequality and theorem.

Castelnuovo’s theorem applies to our original moduli problem, which we take up in §2, where we answer our original problem, at least when  $g$  is large enough. We prove (see Theorem 2.1) that the maximum number of moduli of a linear system of curves of genus  $g \geq 22$  is  $2g + 1$  and it is attained by the linear systems of trigonal canonical curves on a balanced rational normal scroll in  $\mathbb{P}^{g-1}$  (the bound  $g \geq 22$  could be improved, but we thought it useless to dwell on this here). It is remarkable that this maximum is not achieved by linear systems of the largest dimension  $3g + 5$  compatible with a non-trivial map to moduli: indeed, as we said, they consist of hyperelliptic curves, and in fact they dominate the hyperelliptic locus, which has dimension  $2g - 1$  (see Theorem 2.3). The proof of Theorem 2.1 relies on Castelnuovo’s theorem, on the concept of *Castelnuovo pairs*, on their classification and related computation of moduli (see §2.1).

In conclusion, it is worth mentioning, on the same lines as the problem considered here, another more fascinating and complicated one (attributed to F. O. Schreyer): what is, for large enough  $g$ , the maximum dimension of a rational [or, respectively unirational, uniruled, rationally connected] subvariety of  $\mathcal{M}_g$ ?

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## NOTATION, CONVENTIONS AND GENERALITIES

We use standard notation in algebraic geometry. In particular, the symbol  $\equiv$  denotes linear equivalence of divisors. If  $D$  is a divisor on a smooth, projective variety  $X$ ,  $|D|$  is the complete linear system of  $D$ . If  $\mathcal{L}$  is a linear system of divisors on  $X$  of dimension  $r$ ,  $\phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^r$  is the rational map defined by  $\mathcal{L}$ . The system  $\mathcal{L}$  is said to be *simple* if  $\phi_{\mathcal{L}}$  maps  $X$  birationally to its image.

Let  $X$  be a smooth irreducible projective surface. As usual we denote by  $K := K_X$  a *canonical divisor*,  $q := q(X) := h^1(X, \mathcal{O}_X)$  the *irregularity*,  $p_g := p_g(X) := h^0(X, \mathcal{O}_X(K))$  the *geometric genus* of  $X$ .

Let  $D$  be a divisor on  $X$ . We will say that  $D$  is a *curve* on  $X$  if it is effective. If  $D$  is a reduced curve on  $X$ , the *geometric genus*  $g$  of  $D$  is the arithmetic genus of the normalization of  $D$ . Often we will simply call  $g$  the *genus* of  $D$ . We will use the notation  $d = D^2$  and  $r = \dim(|D|)$ . Moreover  $D' \equiv K + D$  is an *adjoint divisor* and  $|D'|$  the *adjoint linear system* to  $D$ . The system  $|D|$  is called *non-special* if it is either empty or  $h^1(X, \mathcal{O}_X(D)) = 0$ .

Suppose there is a morphism  $f : X \rightarrow Y$ , contracting a curve  $C$  of  $X$  to a smooth point  $p$  of a surface  $Y$  and induces an isomorphism between  $X - C$  and  $Y - \{p\}$ . The divisor  $E$ , supported on  $C$ , which is the scheme theoretical fibre of  $f$  over  $p$ , is called a *(-1)-cycle*, or a *(-1)-curve* if  $E = C$  is irreducible.

We will consider pairs  $(X, D)$ , with  $X$  a smooth irreducible projective surface and  $D$  a curve on it. We will extend attributes of  $D$  (like being *nef*, *big*, *ample* etc.) or of  $|D|$  (like being *simple*, *special*, *very ample* etc.) to the pair  $(X, D)$ . We say that  $(X, D)$  is:

- *minimal* if there is no *(-1)-curve*  $C$  on  $X$  such that  $D \cdot C = 0$ ;
- a *h-scroll*, if there is a smooth rational curve  $F$  on  $X$  such that  $F^2 = 0$  and  $D \cdot F = h$ . A 1-scroll will be simply called a *scroll*.

There are obvious notions of morphism, isomorphism, rational and birational maps between pairs (see [5]). We are mainly interested in birational invariants of the linear system  $|D|$  on  $X$ . If  $|D|$  has no fixed curves and its general curve is irreducible, then by blowing up the base locus of  $|D|$  we may assume  $|D|$  is base point free and the general curve of  $D$  is smooth. So we will often assume this is the case. In addition we may assume  $(X, D)$  is minimal by successively contracting all *(-1)-curves*  $E$  with  $D \cdot E = 0$ .

If  $X \cong \mathbb{P}^2$  and  $\ell$  is a line, the pair  $(X, D)$  with  $D \equiv m\ell$  will be called a *m-Veronese pair*.

As usual, we will denote by  $\mathbb{F}_a$  the *Hirzebruch surface*  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ . The Picard group of  $\mathbb{F}_a$  is freely generated by the classes of the divisors:  $E$ , a curve with  $E^2 = -a$  (unique if  $a > 0$ ), and  $F$ , a *ruling*, i.e. a fibre of the structure morphism  $f : \mathbb{F}_a \rightarrow \mathbb{P}^1$ . One has  $F^2 = 0$ ,  $F \cdot E = 1$ . A divisor  $D \equiv \alpha E + \beta F$  is nef as soon as  $D \cdot E = \beta - a\alpha \geq 0$ . If  $\alpha = 1$  and  $\beta \geq a$  then  $\phi_{|D|}$  birationally maps  $\mathbb{F}_a$  to a *rational normal scroll* of degree  $s - 1$  in  $\mathbb{P}^s$ , with  $s = 2\beta - a + 1$ . A pair  $(X, D)$  with  $X \cong \mathbb{F}_a$  and  $D \equiv 2E + (a + g - 1)F$  is nef, the general curve in  $|D|$  is smooth of genus  $g$  and  $r = 3g + 5$ . Such a pair is called a *(a, g)-Castelnuovo pair* (see [12]).

## 1. CASTELNUOVO'S THEOREM

**1.1. Castelnuovo–Enriques theorem.** We recall the following theorem which extends results of Castelnuovo [6] and Enriques [13] (see [12, Theorem 7.3] and [12] also for classical and recent references):

**Theorem 1.1** (Castelnuovo–Enriques theorem). *Let  $(X, D)$  be a pair with  $D$  an irreducible curve. Assume  $d > 0$  and  $(X, D)$  not a scroll. Then:*

$$d \leq 4g + 4 + \epsilon \tag{1}$$

where  $\epsilon = 1$  if  $g = 1$  and  $\epsilon = 0$  if  $g \neq 1$ . Consequently one has:

$$r \leq 3g + 5 + \epsilon \tag{2}$$

and the equality holds in (1) if and only if it holds in (2).

If, in addition, the pair  $(X, D)$  is minimal, then the equality holds in (2), if and only if one of the following happens:

- (i)  $g = 0$ ,  $r = 5$ , and  $(X, D)$  is a 2-Veronese pair;
- (ii)  $g = 1$ ,  $r = 9$ , and  $(X, D)$  is a 3-Veronese pair;
- (iii)  $g = 3$ ,  $r = 14$ , and  $(X, D)$  is a 4-Veronese pair;
- (iv)  $(X, D)$  is a  $(2, n + g + 1)$ -Castelnuovo pair on  $X \cong \mathbb{F}_n$ ,  $n \geq 0$ .

**1.2. Castelnuovo's inequality.** In this section we prove a result of Castelnuovo [7], which specifies (1).

We consider here minimal pairs  $(X, D)$  with  $p_g = q = 0$ ,  $D$  an irreducible, smooth curve of genus  $g \geq 2$ , with  $d \geq 1$  and  $r \geq 1$ , hence  $D$  is nef. By [12, Proposition 7.1], an adjoint curve  $D' \equiv K + D$  is nef and

$$d \leq 4(g-1) + K^2 \leq 4g + 5 \quad (3)$$

which is basically the proof of (1) (in the last inequality we used Miyaoka–Yau inequality). Moreover  $\dim(|D'|) = g - 1$ . Set  $|D'| = P + |M|$ , where  $P$  is the fixed divisor and  $|M|$  is the movable part, called the *pure adjoint system* of  $D$ . We set  $g' := p_a(M)$  and  $d' = M^2$ . One has  $M \cdot D = 2g - 2$  and  $P \cdot D = 0$  and for all curves  $E \leq P$  one has  $E^2 < 0$ .

**Lemma 1.1.** *In the above setting, if  $d \geq 5$  and  $|D|$  is non-special, then  $P = 0$ .*

*Proof.* Reider's Theorem (see [4, 17]) implies that, if  $x$  is a base point of  $|D'|$ , there is an irreducible curve  $A$  containing  $x$ , such that either  $A \cdot D = 1, A^2 = 0$  or  $A \cdot D = 0, A^2 = -1$ .

Let  $E$  be an irreducible curve contained in  $P$ . For all  $x \in E$ , we have a curve  $A_x$  as above. If  $A_x \cdot D = 1$  then  $A_x \neq E$ . Moreover  $A_x^2 = 0$ , so  $A_x$  moves in a base point free pencil  $|A|$ . Since  $A \cdot D = 1$ , we would have  $g = 0$ , a contradiction. Hence  $A_x \cdot D = 0$  and  $E = A_x$ . This shows that  $E^2 = -1$ .

Since  $D \cdot E = 0$  and  $r \geq 1$ , then  $D - E$  is effective. We have the exact sequence  $0 \rightarrow \mathcal{O}_X(D - E) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_E(D) \cong \mathcal{O}_E \rightarrow 0$ , which yields the exact sequence  $H^1(X, \mathcal{O}_X(D)) \rightarrow H^1(E, \mathcal{O}_E) \rightarrow H^2(X, \mathcal{O}_X(D - E))$ . Since  $p_g = 0$ , the last space is 0, and the first is 0 by assumption. Hence  $h^1(E, \mathcal{O}_E) = 0$ , then  $E$  is rational.

In conclusion,  $E$  is a  $(-1)$ -curve such that  $D \cdot E = 0$ , contradicting the minimality assumption.  $\square$

If  $|M|$  is composed with a pencil  $|L|$ , then  $|M| = |(g-1)L|$ ,  $\dim(|L|) = 1$  and  $|L|$  has no base points on  $D$ . Then  $D \cdot L = 2$ ,  $D$  is hyperelliptic and:

- (1) either  $|D|$  cuts out a base point free  $g_2^1$  on the general curve  $L$  of  $|L|$ , hence there is a birational involution  $\iota : X \dashrightarrow X$  that fixes all curves in  $|D|$ , which then is not simple (in this case we say that  $|D|$  is *composed* with the involution  $\iota$ );
- (2) or  $|D|$  cuts out a  $g_2^2$  on  $L$  and  $|L|$  is a pencil of curves of genus 0.

If  $d \geq 5$ , the index theorem implies  $L^2 = 0$ .

**Theorem 1.2** (Castelnuovo's inequality). *Let  $(X, D)$  be minimal with  $D$  smooth and irreducible, with  $g \geq 2$ ,  $d \geq 1$  and  $r \geq 1$ . Assume that either  $D$  is not hyperelliptic or  $|D|$  is not composed with an involution of  $X$ . Then*

$$d \leq 3g + 7 - g' \quad (4)$$

and equality holds if and only if  $X = \mathbb{P}^2$ .

*Proof.* Suppose first  $|M|$  is composed with a pencil  $|L|$ . Then  $D$  is hyperelliptic with  $D \cdot L = 2$  and  $|D|$  is not composed with an involution of  $X$ . Thus the curves in  $|L|$  have genus 0, so  $X$  is rational,  $L^2 = 0$  and  $g' = 2 - g$ . By (3) we have  $d \leq 4g + 5 = 3g + 7 + g - 2 = 3g + 7 - g'$ . If equality holds then  $K^2 = 9$  hence  $X = \mathbb{P}^2$ .

Suppose next  $|M|$  is not composed with a pencil, hence  $d' > 0$ . We have

$$h^0(M, \mathcal{O}_M(M)) = h^0(X, \mathcal{O}_X(M)) - 1 = h^0(X, \mathcal{O}_X(K + D)) - 1 = g - 1$$

then  $|\mathcal{O}_M(M)| = g_{d'}^{g-2}$ . We have two cases: (a)  $K \cdot M \geq 0$ ; (b)  $K \cdot M < 0$

In case (a), since  $D'$  is nef, one has

$$(K + D)^2 \geq (K + D) \cdot M \geq D \cdot M \geq 2g - 2 \quad (5)$$

and equality implies  $K \cdot M = 0$ . Let  $R \equiv D - M \equiv P - K$ . We have two subcases: (a1)  $|R| = \emptyset$ ; (a2)  $R$  is effective. In case (a1), one has  $1 > \chi(\mathcal{O}_X(R))$ , which reads  $d < 3g - g' - 2$  and (4) holds.

In case (a2), one has  $2g - 2 = (K + D) \cdot D = (K + D) \cdot (M + R) \geq (K + D) \cdot M = (K + M + R) \cdot M \geq 2g' - 2$ , then  $g \geq g'$  and, by (5),  $g + g' - 2 \leq 2g - 2 \leq (K + D)^2 = 4g - 4 + K^2 - d$ , therefore (4) holds. If equality holds then  $K^2 = 9$  and  $K \cdot M = 0$  (because equality holds in (5)). This cannot happen on a surface of general type because  $d' > 0$ , thus  $X \cong \mathbb{P}^2$ .

In case (b) one has  $d' > 2g' - 2$ . Then  $g_{d'}^{g-2}$  is not special, hence  $g - 1 = h^0(M, \mathcal{O}_M(M)) = d' - g' + 1$ . Since  $K + D \equiv P + M$  is nef, then  $g + g' - 2 = d' = M^2 \leq (K + D)^2 = 4g - 4 + K^2 - d$ , so  $d \leq 3g - g' + (K^2 - 2) \leq 3g + 7 - g'$  and if equality holds, then  $K^2 = 9$  and, as above,  $X \cong \mathbb{P}^2$ .

Finally, if  $X = \mathbb{P}^2$  then (4) holds with equality.  $\square$

**1.3. Castelnuovo's theorem.** In this section we prove a theorem of Castelnuovo stated in [7] which classifies linear systems on rational surfaces with  $r > g$ . A remark is in order.

*Remark 1.1.* Consider a pair  $(X, D)$  with  $D$  smooth, irreducible such that  $r > g$ . Then  $h^0(D, \mathcal{O}_D(D)) > g$ , hence  $h^1(D, \mathcal{O}_D(D)) = 0$ . Thus  $d - g + 1 = h^0(D, \mathcal{O}_D(D)) > g$  and therefore  $d \geq 2g$ , so that  $K \cdot D < 0$ , which implies that  $X$  has negative Kodaira dimension. This shows that the rationality assumption on  $X$  in Theorem 1.3 below is no restriction. In this case, one has  $h^1(X, \mathcal{O}_X(D)) = 0$ , i.e.  $|D|$  is non-special.

**Theorem 1.3** (Castelnuovo's theorem). *Let  $(X, D)$  be minimal with  $D$  smooth, irreducible, of genus  $g \geq 2$ ,  $X$  rational,  $|D|$  is not composed with an involution of  $X$  and*

$$r \geq \tau(\mu, g) := \frac{(\mu + 2)g + \epsilon_\mu}{\mu} + 2\mu + 3 \quad (6)$$

where

$$\epsilon_\mu = \begin{cases} 1 & \text{for } \mu \text{ odd} \\ 2 & \text{for } \mu \text{ even} \end{cases}$$

Then:

- (i) *either there is a birational morphism  $\phi : X \rightarrow \mathbb{P}^2$  such that  $|D|$  is the proper transform of a linear system of plane curves of degree  $m \leq 2\mu + 1$  with base points of multiplicity  $k \leq \lceil \frac{\mu}{2} \rceil - 1$ ,*
- (ii) *or  $(X, D)$  is a  $m$ -scroll with  $m \leq \mu$ , and precisely there is a birational map  $\phi : X \dashrightarrow \mathbb{F}_a$ , for some  $a \geq 0$ , such that  $|D|$  is the proper transform of a linear system of  $m$ -secant curves to the ruling of  $\mathbb{F}_a$ , with base points of multiplicity  $k \leq \lceil \frac{\mu}{2} \rceil - 1$ .*

*Proof.* Since  $\tau(\mu, g) > g$ , Remark 1.1 applies.

For  $\mu = 1$  one has  $\tau(1, g) = 3g + 6$  and the assertion follows by Castelnuovo-Enriques Theorem 1.1. So we may assume  $\mu \geq 2$ .

If the general curve in  $|M|$  is reducible, then  $|M|$  is composed with a pencil  $|L|$  of curves of genus 0 such that  $L \cdot D = 2$  (see the proof of Castelnuovo's inequality 1.2). Then there is a birational morphism  $\psi : X \rightarrow \mathbb{F}_n$  such that  $|L|$  is the proper transform of the ruling of  $\mathbb{F}_n$ , and  $|D|$  is the proper transform of a linear system of 2-secant curves to the ruling of  $\mathbb{F}_n$ , with at most double base points. By performing elementary transformations based at these double base points, we find case (ii) with  $\mu = 2$ . So from now on we may assume the general curve in  $|M|$  to be irreducible.

Let  $\mu = 2$ . We have  $r = d - g + 1 \geq \tau(2, g) = 2g + 8$ , which is equivalent to  $d \geq 3g + 7$ . By Castelnuovo's inequality  $3g + 7 \leq d \leq 3g + 7 - g'$  then  $g' \leq 0$ . Since  $M$  is irreducible, we have  $g' \geq 0$ , thus  $g' = 0$ , equality holds in (5), hence  $X = \mathbb{P}^2$  and we are in case (i).

Next we assume  $\mu \geq 3$  and we will make induction on  $\mu$ . By Castelnuovo's inequality we have  $r = d - g + 1 \leq 2g - g' + 8$ . If equality holds then  $X \cong \mathbb{P}^2$  and  $(X, D)$  is a  $m$ -Veronese pair, i.e.  $D \in |\mathcal{O}_{\mathbb{P}^2}(m)|$ . We claim that  $m \leq 2\mu + 1$ , i.e. we are in case (i). Indeed  $g = (m - 1)(m - 2)/2$ ,  $r = m(m + 3)/2$  and (6) reads  $2m^2 - 6m(\mu + 1) + 4\mu^2 + 8\mu + 2(\epsilon_\mu + 2) \leq 0$ . The polynomial  $h(x) = 2x^2 - 6x(\mu + 1) + 4\mu^2 + 8\mu + 2(\epsilon_\mu + 2)$  has its critical value at  $x_0 = 3(\mu + 1)/2 < 2\mu$ , so that  $h$  is strictly increasing in  $[x_0, +\infty)$ . If  $m \geq 2\mu + 2$ , we would have  $0 \geq h(m) > h(2\mu + 2) = 2\epsilon_\mu$ , a contradiction.

Now we analyse the case  $r \leq 2g - g' + 7$ . Then  $2g - g' + 7 \geq r \geq \tau(\mu, g)$  implies  $g \geq \frac{\mu g' + \epsilon_\mu}{\mu - 2} + 2\mu$ . Thus

$$\dim(|D'|) = \dim(|M|) = g - 1 \geq \frac{\mu g' + \epsilon_\mu - 2}{\mu - 2} + 2\mu - 1 = \tau(\mu - 2, g'). \quad (7)$$

By induction, we may assume  $r < \tau(\mu - 1, g)$ , hence  $\tau(\mu - 1, g) > \tau(\mu, g)$ , which yields  $g > \mu(\mu - 1) + \frac{1}{2}((\mu - 1)\epsilon_\mu - \mu\epsilon_{\mu-1})$ . Thus

$$g > \begin{cases} \mu(\mu - 1) + \frac{\mu - 2}{2}, & \text{for } \mu \text{ an even number} \\ \mu(\mu - 1) - \frac{\mu + 1}{2}, & \text{for } \mu \text{ an odd number} \end{cases} \quad (8)$$

In particular, if  $\mu \geq 3$ , then  $g \geq 4$  and  $d \geq 2g - 2 \geq 6$ . Then, by Lemma 1.1,  $P = 0$  and  $|D'| = |M|$ .

In view of (7), we would like to apply induction on  $|M|$ , which we can do only if  $M$  verifies the hypotheses of the theorem.

First, we dispose of the case  $g' = 1$ , in which (7) implies  $\dim(|M|) \geq 9$ , and equality holds only for  $\mu = 3$ . Then, by Castelnuovo-Enriques Theorem 1.1,  $\mu = 3$ ,  $(X, M)$  is a 3-Veronese pair and  $D$  is a smooth plane sextic, i.e. we are in case (i). Hence from now on we may assume  $g' \geq 2$ .

*Claim 1.1.* The system  $|M|$  is not composed with an involution of  $X$ .

*Proof of Claim 1.1.* Suppose that  $M$  is composed with a involution  $\iota$  of  $X$ , defined in the Zariski open subset  $U$ . Consider the incidence variety  $\mathcal{V}$  which is the Zariski closure in  $X \times X \times |D|$  of the set  $\{(p, q, D) : p, q \in D, D \in |D|, \iota(p) = q\} \subset U \times U \times |D|$ , with the projections  $\pi_1 : \mathcal{V} \rightarrow X \times X$ ,  $\pi_2 : \mathcal{V} \rightarrow |D|$  to the factors. The image of  $\pi_1$  is the graph  $\Gamma$  of  $\iota$ . Since  $|D|$  is not composed with  $\iota$ , the general fibre of  $\pi_1$  has dimension  $r - 2$ . Hence  $\mathcal{V}$  has an irreducible component  $\mathcal{W}$  which dominates  $\Gamma$  via  $\pi_1$  and has dimension  $r$ .

If  $\pi_{2|\mathcal{W}}$  is surjective, then the general curve in  $|D|$  is hyperelliptic. Since  $\mu \geq 3$  and  $g \geq 4$ , (7) yields  $r \geq 14$  hence  $d \geq 17$ . By Reider's theorem (see again [17, 4]), there is curve  $A$  such that  $0 \leq A \cdot D - 2 \leq A^2 < \frac{A \cdot D}{2} < 2$ . The index theorem implies  $A^2 = 0$ , then  $A \cdot D = 2$  and there is a base point free pencil  $|A|$  which cuts the  $g_2^1$  on the general curve of  $|D|$ . Since  $|D|$  is not composed with an involution, the curves in  $|A|$  have genus 0 (see the discussion before Theorem 1.2). Then  $0 = A \cdot (K + D) = A \cdot M$ . Since the general curve in  $|M|$  is irreducible and  $\dim(|M|) = g - 1 \geq 3$ , this is a contradiction.

If  $\dim(\pi_2(\mathcal{W})) < r$ , let  $D \in \text{Im}(\pi_2)$  be a general element. The general fibre of  $\pi_2$  has dimension at most 1, then it has dimension one, hence  $\dim(\pi_2(\mathcal{W})) = r - 1$ . Now repeat the same argument as above.  $\square$

The pair  $(X, M)$  could be not minimal. If  $E$  is a  $(-1)$ -curve such that  $E \cdot M = 0$ , then  $E \cdot D = 1$ . By contracting these  $(-1)$ -curves we have a birational morphism  $f : X \rightarrow X'$  and there are irreducible curves  $D'$  and  $M'$  on  $X'$  whose proper transform on  $X$  are  $D$  and  $M$ . The linear system  $|D|$  is the proper transform of the sublinear system of  $|D'|$  formed by the curves passing through the points which are blown-up in  $f : X \rightarrow X'$ .

Finally we may apply induction to the pair  $(X', M')$ , and:

- (i') either there is a birational morphism  $\phi' : X' \rightarrow \mathbb{P}^2$  and  $|M'|$  is the proper transform of a linear system  $|C|$  of curves of degree  $d \leq 2\mu - 3$  with base points of multiplicity  $k \leq \lceil \frac{\mu}{2} \rceil - 2$ ;
- (ii') or there is a birational map  $\phi' : X' \dashrightarrow \mathbb{F}_a$ , and  $|M'|$  is the proper transform of a linear system  $|C|$  of  $m$ -secant curves to the ruling of  $\mathbb{F}_a$  with  $m \leq \mu - 2$  with base points of multiplicity  $k \leq \lceil \frac{\mu}{2} \rceil - 2$ .

In case (i'), consider  $\phi = \phi' \circ f : X \rightarrow \mathbb{P}^2$ . If  $\ell$  is a line in  $\mathbb{P}^2$ , set  $H = \phi^*(\ell)$ . We have  $M \equiv dH - \sum_i k_i E_i$ , where  $E_i$  are  $(-1)$ -cycles contracted by  $\phi$  and  $k_i \leq \lceil \frac{\mu-2}{2} \rceil - 1$ . We have also  $K_X \equiv -3H + \sum_i E_i$ . Then  $D \equiv (d+3) - \sum_i (k_i+1)E_i$ , and we are in case (i). The analysis of (ii') is similar and leads to case (ii).  $\square$

## 2. CASTELNUOVO PAIRS AND THEIR MODULI

Castelnuovo–Enriques Theorem 1.1 classifies minimal pairs  $(X, D)$  for which (7) holds with  $\mu = 1$ . For higher  $\mu$ 's we define the concept of  $\mu$ -Castelnuovo pairs.

**2.1. Castelnuovo pairs.** We will call a pair  $(X, D)$  as in Castelnuovo's Theorem 1.3 a  $\mu$ -Castelnuovo's pair, with  $\mu \geq 2$ , if

$$\tau(\mu - 1, g) > r \geq \tau(\mu, g)$$

which implies that (8) holds (see the proof of Theorem 1.3). If for such a pair case (i) [resp. case (ii)] occurs, we say that it presents the *planar case* [resp. the *scroll case*]. Here we list  $\mu$ -Castelnuovo's pairs for  $2 \leq \mu \leq 4$ . The reader may check the details.

**Proposition 2.1.** *If  $(X, D)$  is a  $\mu$ -Castelnuovo's pair with  $2 \leq \mu \leq 4$  with  $D$  smooth of genus  $g \geq 2$ , then:*

- (i)  $(X, D)$  is either an  $m$ -Veronese pair with

$$\begin{aligned} 4 \leq m \leq 5 & \quad \text{if } \mu = 2 \quad \text{and} \quad r = \lceil \tau(2, g) \rceil \\ 6 \leq m \leq 7 & \quad \text{if } \mu = 3 \quad \text{and} \quad r = \lceil \tau(3, g) \rceil + \eta \\ 8 \leq m \leq 9 & \quad \text{if } \mu = 4 \quad \text{and} \quad r = \lceil \tau(4, g) \rceil + \eta \end{aligned}$$

where, in the last two cases,  $\eta = 0$  if  $m$  is odd and  $\eta = 1$  if  $m$  is even, or  $X \cong \mathbb{F}_1$ , and  $D = (m - 1)E + mF$  with

$$\begin{aligned} m = 6 & \quad \text{if } \mu = 3 \\ m = 8 & \quad \text{if } \mu = 4 \end{aligned}$$

and  $r = \lceil \tau(\mu, g) \rceil$  in both cases.



(ii)  $X \cong \mathbb{F}_a$  and  $D \equiv \mu E + \alpha F$  with

$$\alpha \geq \begin{cases} 4 & \text{if } a = 0 \\ 5 & \text{if } a = 1 \\ a\mu & \text{if } a \geq 2 \end{cases}$$

$$g = \begin{cases} \alpha - a - 1 & \text{if } \mu = 2 \\ 2\alpha - 3a - 2 & \text{if } \mu = 3 \\ 3\alpha - 6a - 3 & \text{if } \mu = 4 \end{cases}$$

and  $r = \lceil \tau(\mu - 1, g) \rceil - 1$ . In particular, for  $\mu = 2$ ,  $(X, D)$  is an  $(a, g)$ -Castelnuovo pair.

**2.2. Number of moduli (I).** Consider a pair  $(X, D)$  with  $D$  irreducible, smooth of genus  $g > 0$ . We denote by  $\mathcal{X}_g$  the set of all these pairs. Given  $(X, D) \in \mathcal{X}_g$ , we have the rational *moduli map*  $\mu_D : |D| \dashrightarrow \mathcal{M}_g$ . The dimension of the image of  $\mu_D$  is called the *number of moduli* of  $(X, D)$ , denoted by  $\mu(X, D)$ .

*Problem 2.1.* Given  $g$ , what is the maximum of  $\mu(X, D)$  as  $(X, D)$  varies in  $\mathcal{X}_g$ ?

One might expect that, the larger the dimension of  $|D|$ , the larger  $\mu(X, D)$ . This is not exactly the case as we will see by looking at  $\mu$ -Castelnuovo's pair with  $2 \leq \mu \leq 4$ .

**2.2.1. Veronese pairs.** Consider a  $m$ -Veronese pair  $(X, D)$ , so that  $g = \binom{m-1}{2}$ . This includes case (i) of Proposition 2.1 with  $\mu(X, D)$  maximal. A classical theorem of M. Noether (see [8, §3]) asserts that two smooth plane curves of degree  $m$  are isomorphic if and only if they are projectively equivalent. As a consequence we have:

**Proposition 2.2.** *If  $(X, D)$  is a  $m$ -Veronese pair with  $m \geq 3$ , then  $\mu(X, D) = g + 3m - 9$ .*

The moduli map is dominant if and only if  $m = 4$ , whereas, for  $m \gg 0$ ,  $\mu(X, D) = o(g)$ .

**2.2.2. Castelnuovo pairs.** Consider an  $(a, g)$ -Castelnuovo pair  $(X, D)$ , with  $g \geq 2$ , which is then a 2-Castelnuovo pair. The curve  $D$  is hyperelliptic hence the image of  $\mu_D$  is contained in the hyperelliptic locus  $\mathcal{H}_g$  in  $\mathcal{M}_g$  and therefore  $\mu(X, D) \leq 2g - 1$ .

**Proposition 2.3.** *If  $(X, D)$  is an  $(a, g)$ -Castelnuovo pair, then  $\mu(X, D) = 2g - 1$ , i.e.  $\text{Im}(\mu_D) = \mathcal{H}_g$ .*

*Proof.* We have  $X = \mathbb{F}_a$  and  $D = 2E + (a + g + 1)F$ . Consider the exact sequence  $0 \rightarrow T_D \rightarrow T_X|_D \rightarrow N_{D,S} \rightarrow 0$ . To prove the assertion it suffices to prove that

$$\dim(\text{Im}(H^0(D, N_{D,X}) \rightarrow H^1(D, T_D))) = 2g - 1. \quad (9)$$

We have  $h^0(D, N_{D,X}) = r = 3g + 5$  and  $h^0(D, T_D) = 0$ . So (9) is equivalent to  $h^0(T_X|_D) = g + 6$ . Consider the structure morphism  $f : X \rightarrow \mathbb{P}^1$  and let  $T_f$  be the relative tangent sheaf. We have the exact sequence  $0 \rightarrow T_f|_D \rightarrow T_X|_D \rightarrow \mathcal{O}_X(2F)|_D \rightarrow 0$ , from which  $\deg(T_f|_D) = 2g + 2$ , hence  $h^1(D, T_f|_D) = 0$  and therefore  $h^0(D, T_X|_D) = h^0(D, T_f|_D) + h^0(D, \mathcal{O}_D(2F)) = g + 6$ , as needed.  $\square$

Next we consider  $\mu$ -Castelnuovo pairs as in part (ii) of Proposition 2.1 with  $3 \leq \mu \leq 4$ . The analysis of the moduli maps in these cases could be done, as in the proof of Proposition 2.3, by studying the coboundary map in (9). There is however a quicker way, which parallels Noether's theorem for plane curves.

**Proposition 2.4.** *Let  $(X, D)$  be a  $\mu$ -Castelnuovo pair as in part (ii) of Proposition 2.1 with  $3 \leq \mu \leq 4$  and  $g \geq 4$ . Then two smooth curves  $C, C' \in |D|$  are isomorphic if and only if there is an automorphism  $\omega$  of  $X \cong \mathbb{F}_a$  such that  $C' = \omega(C)$ . Accordingly*

$$\mu(X, D) = \begin{cases} 2g + 1 & \text{if } \mu = 3 \text{ and } a = 0 \\ 2g + 2 - a & \text{if } \mu = 3 \text{ and } a > 0 \\ \frac{5g}{3} + 3 & \text{if } \mu = 4 \text{ and } a = 0 \\ \frac{5g}{3} + 4 - a & \text{if } \mu = 4 \text{ and } a > 0 \end{cases} \quad (10)$$

In particular, for  $\mu = 3$  and  $0 \leq a \leq 1$ , the image of  $\mu_D$  is the whole trigonal locus.

*Proof.* Consider the case  $\mu = 3$ . Then  $D \equiv 3E + \alpha F$  with  $\alpha \geq 3a$ . Set  $H \equiv D + K = E + (\alpha - a - 2)F$ . Then  $\phi_{|H|}$  is a morphism mapping  $X$  to a rational normal scroll  $S$  in  $\mathbb{P}^{g-1}$ , and the smooth curves in  $|D|$  are mapped to canonical curves. Two of such curves  $C, C'$  are isomorphic if and only if there is a projective transformation  $\omega$  of  $\mathbb{P}^{g-1}$  such that  $C' = \omega(C)$ . Since  $\omega(S) = S$ , the first assertion follows.

Note that  $r = \tau(2, g) - 1 = 2g - 7$ . The automorphisms group of  $\mathbb{F}_a$  has dimension  $a + 5$  if  $a > 0$  and 6 if  $a = 0$ , which explains the first two lines of (10).

Look now at the case  $\mu = 4$ . Assume first  $a \geq 3$ . Then  $D \equiv 4E + \alpha F$  with  $\alpha \geq 4a$ . Set  $H \equiv E + \beta F$ , with  $\beta$  verifying

$$\alpha \leq 4\beta \leq 2\alpha - 2a - 2. \quad (11)$$

Since  $\alpha \geq 4a$  and  $a \geq 3$ , certainly such a  $\beta$  exists. In addition  $\beta \geq \alpha/4 \geq a$ , hence  $\phi_{|H|}$  maps  $X$  to a rational normal scroll in  $\mathbb{P}^s$ , with  $s = 2\beta - a + 1$ . The curves in  $|D|$  map to curves of degree  $n = 4\beta + \alpha - 4a$ . Note that  $n - 1 = 3(s - 1) + \epsilon$ , with  $\epsilon = \alpha - a - 2\beta - 1$ . By (11), one has  $0 \leq \epsilon < s - 1$ . Then the maximal genus of curves of degree  $n$  in  $\mathbb{P}^s$  is  $3\alpha - 6n - 3 = g$  (see, e.g., [14, p. 527]). Hence the smooth curves in  $|D|$  are Castelnuovo curves in  $\mathbb{P}^s$ . By a result of Accola (see [1] and also [8, Teorema (2.11)]), two smooth curves  $C, C' \in |D|$  are isomorphic if and only if they are projectively equivalent in  $\mathbb{P}^s$ . The conclusion is as for  $\mu = 3$ .

In case  $0 \leq a \leq 2$ , the same argument as above applies if  $\alpha \geq 5 + 2a$ , since in this case still there is an integer  $\beta$  verifying (11). So we are left to consider the cases  $0 \leq a \leq 2$  with  $\alpha \leq 4 + 2a$ . Then  $\alpha = 4 + 2a$  by Proposition 2.1, (ii), for  $a = 0, 2$ . In case  $a = 1$  also  $\alpha = 4 + 2a = 6$ , because  $\mu = 4, \alpha = 5$  does not correspond to a 4-Castelnuovo pair. The argument is similar to the above and therefore we will be brief. For  $a = 0, 2$  we map  $\mathbb{F}_a$  to a quadric  $S$  in  $\mathbb{P}^3$ . Then the curves in  $|D|$  are complete intersections of  $S$  with a surface of degree 4. Again two smooth curves  $C, C' \in |D|$  are isomorphic if and only if they are projectively equivalent in  $\mathbb{P}^3$  (see [11] or [8, Corollario (4.8)]). If  $a = 1$  then  $\mathbb{F}_1$  birationally maps to the plane by contracting  $E$  and  $|D|$  is the proper transform of the linear system of curves of degree 6 with a double base point. Two such curves with only one node are birational if and only if they are projectively equivalent in  $\mathbb{P}^2$  (see [8, Osservazione (2.19)]) and the conclusion is as above.  $\square$

The last assertion in Proposition 2.4 is no news: indeed it goes back to Maroni [15].

### 2.3. Number of moduli (II).

In this section we answer Problem 2.1.

**Theorem 2.1.** *Let  $(X, D)$  be a minimal pair with  $D$  a smooth curve of genus  $g \geq 22$ . Then  $\mu(X, D) \leq 2g + 1$  and equality holds if and only if  $(X, D)$  is a 3-Castelnuovo pair with  $X \cong \mathbb{F}_a$  and  $0 \leq a \leq 1$ , in which case the image of  $|D|$  via  $\mu_D$  is the trigonal locus in  $\mathcal{M}_g$ .*

*Proof.* By Remark 1.1, we may assume  $X$  has negative Kodaira dimension, otherwise  $r \leq g$ . If  $q > 0$ , there is a curve  $C$  of genus  $q$  and a surjective morphism  $f : X \rightarrow C$ , hence all curves in  $|D|$  map to  $C$  and therefore  $\mu(X, D) \leq 2g - 2$  (see [10]). So we may assume  $q = 0$ .

If  $|D|$  is composed with an involution, then the general curve  $D \in |D|$  has a non-constant morphism  $D \rightarrow C$  to a curve. If  $C$  is rational, then  $D$  is hyperelliptic and  $\mu(X, D) \leq 2g - 1$ , if  $C$  is irrational, one has  $\mu(X, D) \leq 2g - 2$  as above. Thus, if  $\mu(X, D) \geq 2g$  we may assume that  $(X, D)$  verifies the hypotheses of Castelnuovo's Theorem 1.3.

By Castelnuovo–Enriques Theorem 1.1, we may assume that  $r \leq 3g + 5 = \tau(1, g) - 1$ . If  $r \geq 2g + 8 = \tau(2, g)$ , then  $(X, D)$  is a 2-Castelnuovo pair. By Propositions 2.1, 2.2 and 2.3, the image of  $\mu_D$  is  $\mathcal{H}_g$  and  $\mu(X, D) = 2g - 1$ . If  $\tau(2, g) - 1 \geq r \geq \tau(3, g) = \frac{5g+1}{3} + 9$ , then  $(X, D)$  is a 3-Castelnuovo pair. By Propositions 2.1, 2.2 and 2.4, the maximum of  $\mu(X, D)$  is attained if  $(X, D)$  is as in (ii) of Proposition 2.1 with  $0 \leq a \leq 1$ . In this case  $\mu(X, D) = 2g + 1$  and the image of  $|D|$  via  $\mu_D$  is the trigonal locus. If  $\tau(3, g) > g \geq \tau(4, g) = \frac{3g+1}{2} + 11$ , then the maximum of  $\mu(X, D)$  is  $\frac{5g}{3} + 3$  (see Proposition 2.4) which is smaller than  $2g + 1$  if  $g \geq 7$ . Finally, if  $r < \frac{3g+1}{2} + 11$ , then also  $\mu(X, D) \leq \frac{3g+1}{2} + 11$  and this is smaller than  $2g + 1$  if  $g \geq 22$ .  $\square$

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